Linear-Time Computation of a Linear Kernel for Dominating Set on Planar Graphs

René van Bevern[1]

joint work with

Sepp Hartung[1], Frank Kammer[2], Rolf Niedermeier[1], and Mathias Weller[1]

[1]Institut für Softwaretechnik und Theoretische Informatik
Technische Universität Berlin, Germany

[2]Institut für Informatik
Universität Augsburg, Germany

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Dominating Set

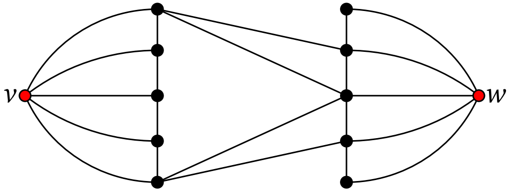
Dominating Set
Input: A graph $G = (V, E)$ and a natural number k.
Question: Is there a dominating set $D \subseteq V$ with $|D| \leq k$ for G?

Here, D is a dominating set $\iff V \subseteq N[D]$.

Size of a minimum dominating set for G is denoted by $\gamma(G)$.

Dominating Set is NP-hard even when restricted to planar graphs.

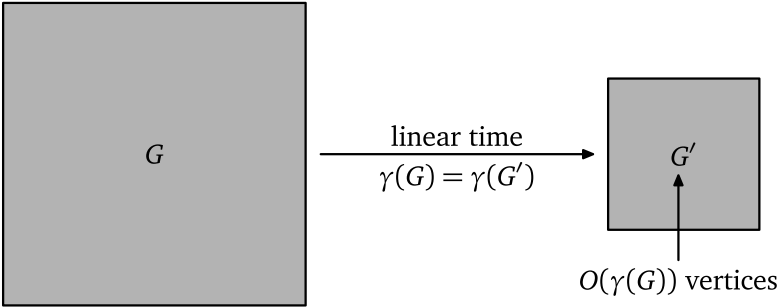
Example for Dominating Set



The set $\{v, w\}$ dominates the shown graph.

Problem Kernels

We show an $O(\gamma(G))$-vertex problem kernel for Dominating Set on planar graphs that is computable in linear time.



Why Problem Kernels Matter

In general, problem kernels yield equivalent smaller instances in polynomial time. Hence, used as a preprocessing step, they can

- speed up algorithms that exactly solve NP-hard problems,
- speed up slow but effective approximation algorithms,
- speed up slow but effective kernelization algorithms,
- speed up heuristics often used in practice.

Combining fast kernelization algorithms with effective ones yields small kernels fast.

In contrast, it is unclear how a fast approximation algorithms can be combined with an effective one to get an approximation algorithm that is both fast and effective.

The Kernel Size Race

Problem kernels are currently the hottest topic in parameterized algorithmics, resulting in a kernel size race.

An Example:

FEEDBACK VERTEX SET: delete at most k vertices to transform a graph into a forest (make it cycle-free).

Burrage, Estivill-Castro, Fellows, and Langston, IWPEC 2006:
$O(k^{11})$-vertex kernel

Bodlaender and van Dijk, TCS 2010:
$O(k^3)$-vertex kernel

Thomassé, ACM Trans. Algorithms 2010:
$O(k^2)$-vertex kernel

The Kernel Size Race

Problem kernels are currently the hottest topic in parameterized algorithmics, resulting in a kernel size race.

Another Example:

Cluster Editing: delete or add at most k edges to transform a graph into a disjoint union of cliques.

Gramm, Guo, Hüffner, and Niedermeier, TCS 2005:
$O(k^2)$-vertex kernel

Fellows, Langston, Rosamond, and Shaw, FCT 2007:
$O(k)$-vertex kernel

Guo, TCS 2009:
$4k$-verex kernel

Chen and Meng, COCOON 2010:
$2k$-vertex kernel

Problem Kernels for Dominating Set

Focusing on Kernel Size:

Alber, Fellows, and Niedermeier, J. ACM, 2004:
335γ-vertex kernel in $O(n^3)$ time on planar graphs.

Chen, Fernau, Kanj, and Xia, SIAM J. Comput., 2007:
67γ-vertex kernel in $O(n^3)$ time on planar graphs.

Focusing on Larger Graph Classes:

Fomin and Thilikos, ICALP 2004:
$O(\gamma + g)$-vertex kernel in $O(gn^3)$ time on graphs of genus g.

Philip, Raman, and Sikdar, ESA 2009:
$O(\gamma^{2(d+1)^2})$-vertex kernel in $O(2^d dn^2)$ time on d-degenerate graphs.

Fomin, Lokshtanov, Saurabh, and Thilikos, SODA 2010:
$O(\gamma)$-vertex kernel in polynomial time on apex-minor free graphs.

Linear-Time Problem Kernel for Dominating Set

Focusing on Running Time:

van Bevern, Hartung, Kammer, Niedermeier, Weller, Manuscript 2011 (submitted):
O(γ)-vertex kernel in O(n) time.

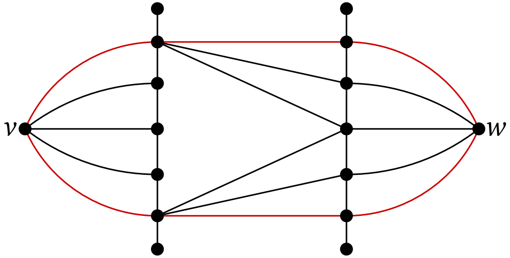
Small Kernel Fast:

67γ-vertex kernel in O($\gamma^3 + n$) time using the above kernelization as preprocessing step for the following result:

Chen, Fernau, Kanj, Xia, SIAM J. Comput., 2007:
67γ-vertex kernel in O(n^3) time on planar graphs.

Tools for Obtaining Dominating Set Problem Kernels

Many results exploit a result by Alber, Fellows, and Niedermeier, J. ACM 2004: a graph is decomposable into $O(\gamma)$ *regions*:

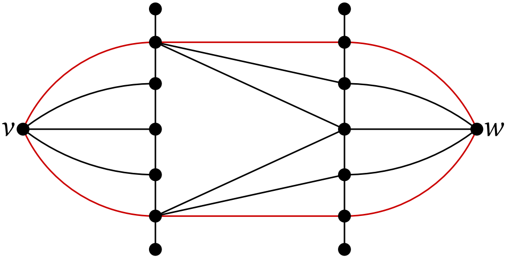


A *region* $R(v, w)$ between two vertices v and w is a closed bounded subset of the plane such that:

1. the boundary of $R(v, w)$ is formed by two simple paths between v and w, each of which has length at most three and
2. all vertices inside the region $R(v, w)$ are from $N[v] \cup N[w]$.

Problem Kernel Analysis of Alber et al., J. ACM, 2004

A graph is decomposable into $O(\gamma)$ regions:



Problem kernel is obtained as follows:

1. shrink each region to $O(1)$ vertices and
2. reduce number of vertices outside of regions to $O(\gamma(G))$.

Our Problem Kernel

Re-use the region framework and data reduction rules by Alber et al. with slight modifications, but find and update the structures to be reduced in linear time.

Example: Finding and Shrinking Regions

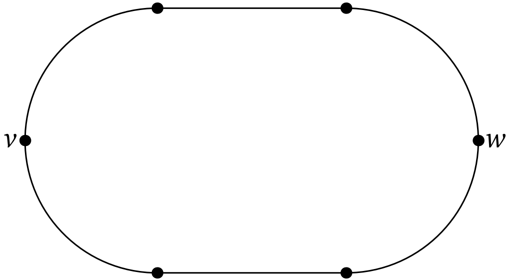
Problem: graph is decomposable into $O(\gamma)$ regions, but we do not actually *have* them $\rightsquigarrow$ shrink everything that *could* be a region.

Alber et al. delete vertices from $N[v] \cup N[w]$ for all vertices v, w.
$\rightsquigarrow$ $\Omega(n^2)$ time. Rules applied $\mathrm{O}(n)$ times $\rightsquigarrow$ $\mathrm{O}(n^3)$ time.

Assuming that we know how to shrink regions: how to find them?

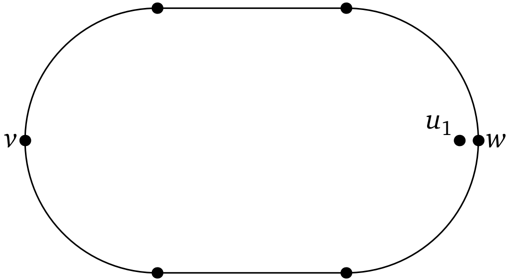
- Largely exploit the restricted structure of regions and
- know when to stop: do not apply data reduction rules more often than necessary to obtain the problem kernel.

Shrinking Regions in Linear Time



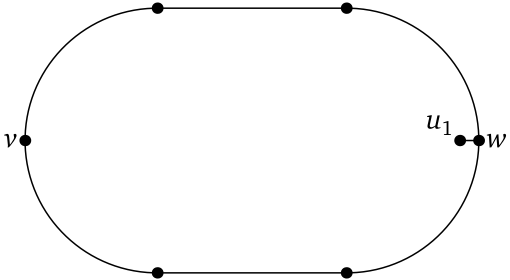
Consider this region $R(v, w)$.

Shrinking Regions in Linear Time



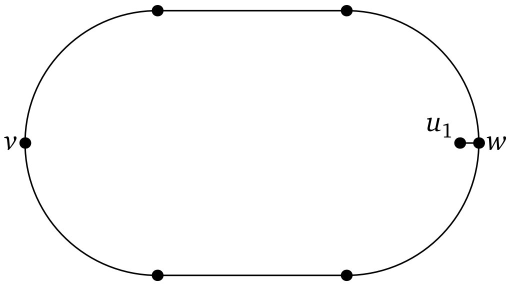
Assume that $u_1 \in R(v, w)$. Then, $u_1 \in N[v, w]$.

Shrinking Regions in Linear Time



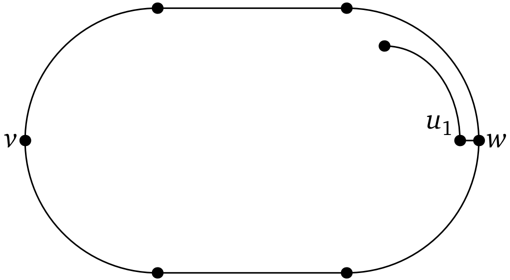
W. l. o. g., assume that $u_1 \in N(w)$.

Shrinking Regions in Linear Time



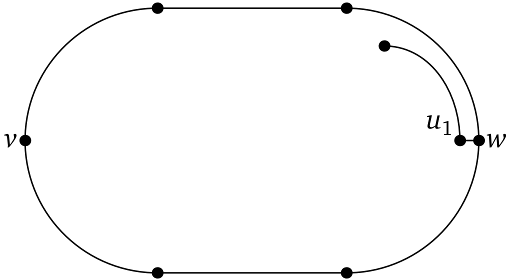
If $\deg(u_1) \leq 1$: delete u_1 and remember w to be dominating.

Shrinking Regions in Linear Time



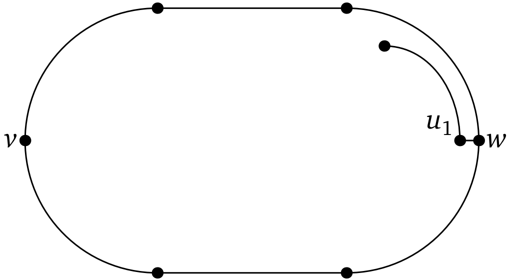
Hence, u_1 has a neighbor, which, due to planarity, is in $R(v, w)$.

Shrinking Regions in Linear Time



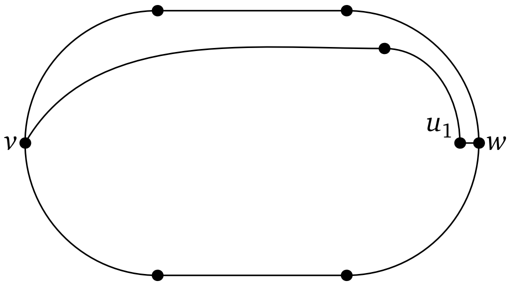
If $N[N[u_1]] \subseteq N[w]$, delete u_1 (remember w to be dominating).

Shrinking Regions in Linear Time



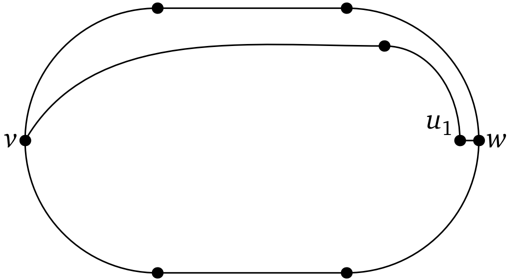
Assume that u_1's neighbor is nonadjacent to w.

Shrinking Regions in Linear Time



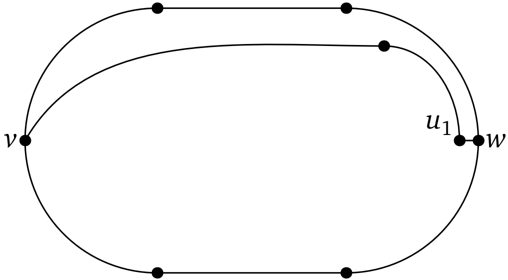
Then, u_1's neighbor is adjacent to v.

Shrinking Regions in Linear Time



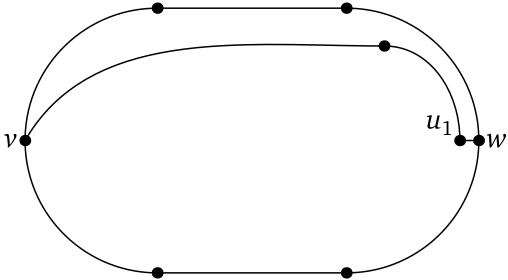
Observation: inner vertices of $R(v, w)$ lie on short v-w-paths.

Shrinking Regions in Linear Time



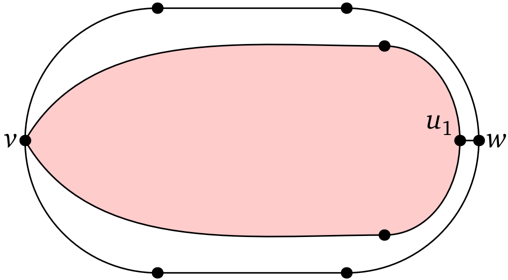
Can quickly list such paths if vertices of $R(v, w)$ have small degree.

Shrinking Regions in Linear Time



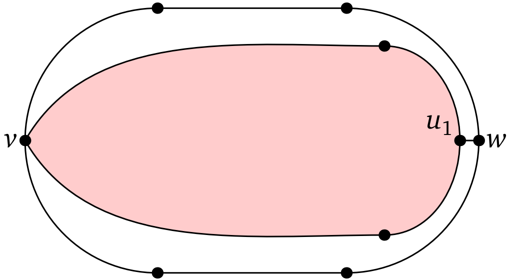
If u_1 has large degree, then $N(u_1) \cap N(v)$ or $N(u_1) \cap N(w)$ is large.

Shrinking Regions in Linear Time



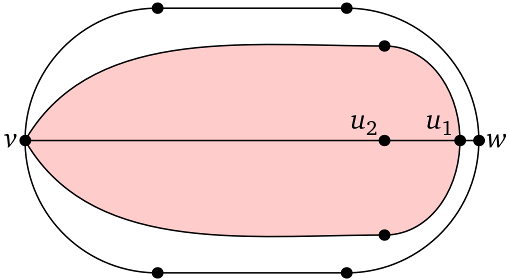
Assume that $N(u_1) \cap N(v)$ is large ⇝ red sub-region $R(v, u_1)$.

Shrinking Regions in Linear Time



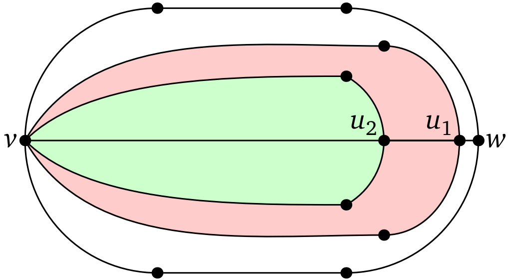
Can shrink $R(v, u_1)$ if it only contains vertices with small degree.

Shrinking Regions in Linear Time



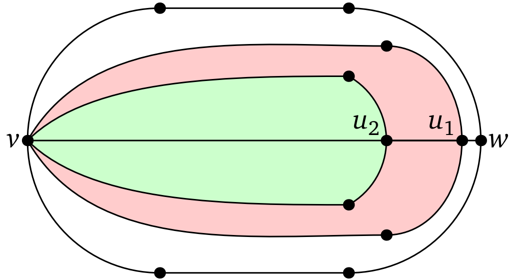
Assume that $u_2 \in R(v, u_1)$. By planarity, $|N(u_2) \cap N(1)| \in O(1)$.

Shrinking Regions in Linear Time



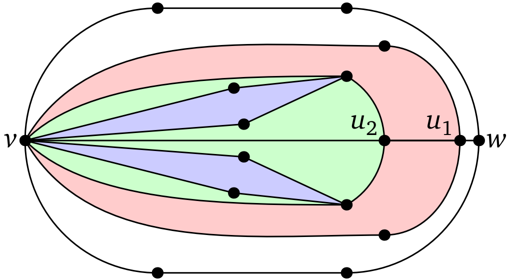
Hence, if u_2 has large degree, then $N(u_2) \cap N(v)$ (green) is large.

Shrinking Regions in Linear Time



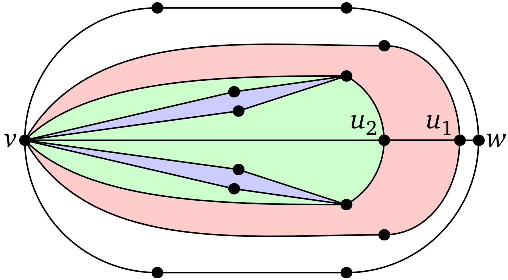
Can shrink $R(v, u_2)$ if it only contains vertices with small degree.

Shrinking Regions in Linear Time



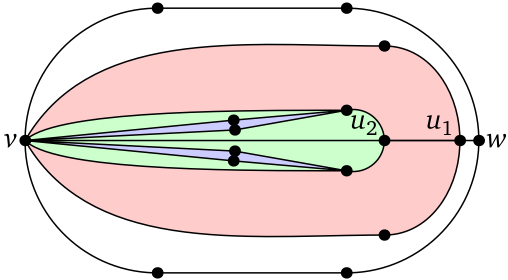
This is the case since vertices in blue regions are safely deletable.

Shrinking Regions in Linear Time



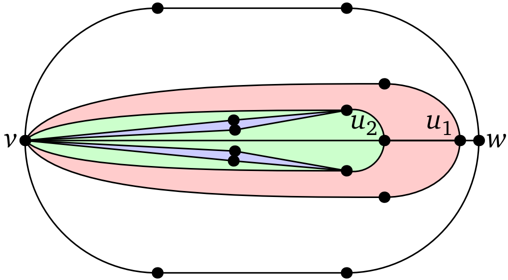
Shrink blue regions ⇝ vertices in $R(v, u_2)$ have low degree.

Shrinking Regions in Linear Time



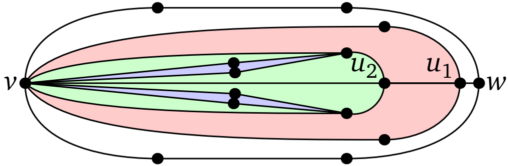
Shrink $R(v, u_2) \rightsquigarrow$ vertices in $R(v, u_1)$ have low degree.

Shrinking Regions in Linear Time



Shrink $R(v, u_1) \rightsquigarrow$ vertices in $R(v, w)$ have low degree.

Shrinking Regions in Linear Time



We can now find and shrink $R(v, w)$.

Shrinking Regions in Linear Time

Three times, basically apply the following algorithm:

1. Preprocess (e. g. degree-one deletion rule)
2. Find deletion candidates regions:
   - **2.1** For each $v \in V$, in $O(\deg(v))$ time, list all short paths starting at v, having length at most four, and only crossing vertices with constant degree.
   - **2.2** For each such path p, let v_p, w_p be its endpoints.
   - **2.3** If p contains only vertices of $N[v_p] \cup N[w_p]$, then p is a deletion candidate (could be part of a region to shrink).
3. Shrink regions, which contain $O(n)$ deletion candidates in total, since only $\sum_{v \in V} \deg(v) = O(n)$ paths are generated.

Conclusion

Not only the size of problem kernels, but also the running time of kernelization algorithms has to be optimized.

Optimize one of size and speed; get the other for free:

- can combine speed-optimized and size-optimized kernelization algorithms to obtain small kernels fast

⇝ 67γ-vertex kernel in $O(\gamma^3 + n)$ time for DOMINATING SET in planar graphs.[1]

⇝ faster algorithms for solving hard problems.

[1] Executing the algorithm by Chen et al. (SIAM J. Comput., 2007) after ours